

WALDSCHMIDT CONSTANT OF THREE FAT COORDINATE POINTS IN $\mathbb{P}^2$

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ABSTRACT

The Waldschmidt constant, introduced by Waldschmidt in 1975, has become an important invariant in commutative algebra and algebraic geometry. Much of the existing work has focused on establishing bounds and studying its asymptotic behavior. Nevertheless, explicit computations of Waldschmidt constants remain difficult, even for relatively simple configurations of points in the projective plane. In this paper, we determine the Waldschmidt constant of the ideal defining three fat coordinate points in $\mathbb{P}^2$. Our approach is based on describing the symbolic polyhedron of the ideal and expressing the computation of its Waldschmidt constant as a linear programming problem over this polyhedron.

Keywords: Fat point; symbolic polyhedron; symbolic power; Waldschmidt constant.

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1. INTRODUCTION

In the past decade, there has been significant interest in the "ideal containment problem": given a nontrivial homogeneous ideal I of a polynomial ring $R = \mathbb{k}[x_1, \dots, x_n]$ over a field $\mathbb{k}$, the goal is to determine all pairs of positive integers (m, r) such that $I^{(m)} \subseteq I^r$. Here, $I^{(m)}$ denotes the m -th symbolic power of I , and I^r is the ordinary r -th power of I . Recall that the m -th symbolic power of I is defined as

$$I^{(m)} = \bigcap_{\mathfrak{p} \in \text{Ass}(I)} (I^m R_{\mathfrak{p}} \cap R),$$

where I^m is the ordinary m -th power of I , $\text{Ass}(I)$ denotes the set of associated primes of I , and $R_{\mathfrak{p}}$ is the localization of R at the prime ideal $\mathfrak{p}$.

This problem was motivated by the fundamental results of [9], which show that containment holds whenever $m \geq r(n-1)$. To capture more precise information about these containments, Bocci and Harbourne [1] introduced the *resurgence* of I , denoted $\rho(I)$, defined as

$$\rho(I) = \sup \left\{ \frac{m}{r} \mid I^{(m)} \not\subseteq I^r \right\}.$$

In general, computing $\rho(I)$ is quite difficult. Since [1], there has been ongoing research to bound $\rho(I)$ in terms of other invariants of I that may be easier to compute. One such bound involves the Waldschmidt constant of I . Given any nonzero homogeneous ideal I of R , let $\alpha(I) = \min\{d \mid I_d \neq 0\}$ be the smallest degree of a nonzero element in I . The *Waldschmidt constant* of I is then defined as

$$\hat{\alpha}(I) = \lim_{m \rightarrow \infty} \frac{\alpha(I^{(m)})}{m}.$$

This limit exists and was first introduced by Waldschmidt [11] for ideals of finite point sets in the context of complex analysis. In the language of projective varieties, Waldschmidt was interested in determining the minimal degree of a hypersurface passing through a collection of points with prescribed multiplicities. Specifically, he was interested in determining $\alpha(I^{(m)})$ when I defines a set of points.

Bocci and Harbourne [1] showed that $\frac{\alpha(I)}{\hat{\alpha}(I)} \leq \rho(I)$. Therefore, computing or bounding $\hat{\alpha}(I)$ is of significant interest. Dumnicki [5] found lower bounds for $\hat{\alpha}(I)$ when I is the ideal of generic points in $\mathbb{P}^3$, while Dumnicki et al. [6] computed $\hat{\alpha}(I)$ for ideals of points defined by hyperplane arrangements. Guardo et al. [7] also computed $\hat{\alpha}(I)$ when I is the ideal of general sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$. Although computing $\hat{\alpha}(I)$ may be easier than computing $\rho(I)$, the problem of computing the Waldschmidt constant remains challenging in general.

This paper focuses on computing $\hat{\alpha}(I)$ when I defines three fat coordinate points in $\mathbb{P}^2$ with multiplicities m_1, m_2, m_3 , that is,

$$I = (x, y)^{m_1} \cap (y, z)^{m_2} \cap (z, x)^{m_3}.$$

We derive a closed formula for $\hat{\alpha}(I)$, which is determined by a simple optimization problem involving the three multiplicities. Specifically, we prove the following.

Theorem 1.1. *Let $I = (x, y)^{m_1} \cap (y, z)^{m_2} \cap (z, x)^{m_3} \subseteq \mathbb{k}[x, y, z]$. Then*

$$\hat{\alpha}(I) = \max \left\{ m_1, m_2, m_3, \frac{m_1 + m_2 + m_3}{2} \right\}.$$

The paper is organized as follows. In Section 2, we recall the basic notions and results concerning Newton polyhedra, symbolic polyhedra, Waldschmidt constants, and symbolic analytic spread for monomial ideals. In Section 3, we determine the vertices of the symbolic polyhedron associated with the ideal defining three fat coordinate points, derive an explicit formula for its Waldschmidt constant, and compute its symbolic analytic spread.

2. SYMBOLIC POLYHEDRON OF A MONOMIAL IDEAL

Throughout the section, $\mathbb{k}$ denotes a field, and $R = \mathbb{k}[x_1, \dots, x_n]$ is a polynomial ring over $\mathbb{k}$. For $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{N}^n$, let $x^{\mathbf{a}}$ denote the monomial $x_1^{a_1} \cdots x_n^{a_n}$ in R .

Definition 2.1. ([10, Definition 1.4.7]) For a monomial ideal I , the Newton polyhedron of I , denoted $\text{NP}(I)$, is the convex hull of the exponent vectors for all the monomials in I :

$$\text{NP}(I) = \text{convex hull}\{a \in \mathbb{N}^n \mid x^a \in I\}.$$

One of the useful properties of Newton polyhedra is that they scale linearly upon taking ordinary powers of ideals, namely the following identity holds for all $m \in \mathbb{N}$:

$$\text{NP}(I^m) = m \text{NP}(I).$$

Let $\text{maxAss}(I)$ denote the set of maximal associated primes of I .

Definition 2.2. ([3]) The *symbolic polyhedron* of a monomial ideal I with primary decomposition $I = Q_1 \cap \cdots \cap Q_s$ is

$$\text{SP}(I) = \bigcap_{\mathfrak{p} \in \text{maxAss}(I)} \text{NP}(Q_{\subseteq \mathfrak{p}}),$$

where for each $\mathfrak{p} \in \text{maxAss}(I)$, denote $Q_{\subseteq \mathfrak{p}} = \bigcap_{\sqrt{Q_i} \subseteq \mathfrak{p}} Q_i$.

Remark 2.3. The ideals $Q_{\subseteq \mathfrak{p}}$ are uniquely determined by I and $\mathfrak{p}$ and are independent of the primary decomposition of I . This follows from the identity $Q_{\subseteq \mathfrak{p}} = IR_{\mathfrak{p}'} \cap R$, where $\mathfrak{p}'$ is the prime monomial ideal generated by the variables of R that are not in $\mathfrak{p}$. Consequently, the symbolic polyhedron $\text{SP}(I)$ does not depend on the choice of primary decomposition.

Definition 2.4. [8] We say that a monomial ideal I is of *linear-power type* if I can be written as

$$I = \bigcap_{\mathfrak{p} \in \text{Min}(I)} \mathfrak{p}^{\omega_{\mathfrak{p}}},$$

in which $\omega_{\mathfrak{p}} \in \mathbb{N}$ and all minimal primes of I are monomial primes.

Proposition 2.5. [4, 8] Let $I \subseteq R$ be a monomial ideal of linear-power type and let $\mathbf{a} \in \mathbb{Z}_{\geq 0}^n$.

1. For any given integer k , we have

$$x^{\mathbf{a}} \in I^{(k)} \text{ if and only if } \frac{\mathbf{a}}{k} \in \text{SP}(I).$$

2. Assume that $I = \bigcap_{\mathfrak{p} \in \text{Min}(I)} \mathfrak{p}^{\omega_{\mathfrak{p}}}$, we have

$$\text{SP}(I) = \bigcap_{\mathfrak{p} \in \text{Min}(I)} \{(a_1, \dots, a_n) \in \mathbb{R}_{\geq 0}^n \mid \sum_{x_i \in \mathfrak{p}} a_i \geq \omega_{\mathfrak{p}}\}.$$

Example 2.6. Let $I = (xy, yz, zx) = (x, y) \cap (y, z) \cap (z, x) \subseteq R = \mathbb{k}[x, y, z]$. It can be seen that

$$\text{NP}(I) = \text{convex hull}\{(1, 1, 0), (1, 0, 1), (0, 1, 1)\} + \mathbb{R}_{\geq 0}^3.$$

By the description of $\text{SP}(I)$, the defining half-spaces for $\text{SP}(I)$ are: $a_1 \geq 0, a_2 \geq 0, a_3 \geq 0, a_1 + a_2 \geq 1, a_2 + a_3 \geq 1$ and $a_3 + a_1 \geq 1$. Thus,

$$\text{SP}(I) = \text{convex hull}\left\{(1, 1, 0), (1, 0, 1), (0, 1, 1), \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)\right\} + \mathbb{R}_{\geq 0}^3.$$

It is obvious from that $\text{NP}(I) \neq \text{SP}(I)$.

The initial degree of a monomial ideal I can be expressed as the solution of a linear programming problem in the following manner:

$$\alpha(I) = \min\{y_1 + \cdots + y_n \mid (y_1, \dots, y_n) \in \text{NP}(I)\}.$$

This is because the optimal solution is attained at a vertex of $\text{NP}(I)$ and the vertices of $\text{NP}(I)$ correspond to a subset of the minimal generators of I .

Building on this, we now turn to the Waldschmidt constant of a monomial ideal I , which can be framed as a similar linear optimization problem. The feasible region for this problem is given by the symbolic polyhedron $\text{SP}(I)$. More precisely, we have the following result.

Proposition 2.7. [2, Corollary 4.6] *The Waldschmidt constant of a monomial ideal I is the solution to the following linear optimization problem with feasible region given by its symbolic polyhedron:*

$$\begin{aligned} &\text{minimize} && y_1 + \cdots + y_n \\ &\text{subject to} && (y_1, \dots, y_n) \in \text{SP}(I). \end{aligned}$$

Example 2.8. Let $I = (xy, xz, yz)$ as in Example 2.6. The Newton polyhedron $\text{NP}(I)$ has the vertices $(1, 1, 0)$, $(1, 0, 1)$, $(0, 1, 1)$, and hence $\alpha(I) = 2$.

The symbolic polyhedron $\text{SP}(I)$ has vertices at $(1, 1, 0)$, $(1, 0, 1)$, $(0, 1, 1)$ and $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. By Proposition 2.7, one has $\hat{\alpha}(I) = \frac{3}{2}$.

Finally, we establish a connection between the symbolic polyhedron and the symbolic analytic spread of a monomial ideal.

Definition 2.9. [9] Let I be an ideal of R . The *symbolic Rees algebra* of I is defined by

$$\mathcal{R}_s(I) = \bigoplus_{k \geq 0} I^{(k)} t^k \subseteq R[t].$$

Definition 2.10. ([8]) Let I be an ideal and let $\mathfrak{m} = (x_1, \dots, x_n)$ be the maximal homogeneous ideal in R . The *symbolic analytic spread* of I is defined to be

$$\ell_s(I) = \dim(\mathcal{R}_s(I)/\mathfrak{m}\mathcal{R}_s(I)) = \dim(\mathcal{R}_s(I) \otimes_R R/\mathfrak{m}).$$

The following result establishes a connection between the symbolic analytic spread and the geometry of the symbolic polyhedron.

Proposition 2.11. [8, Corollary 4.5] *Let $I \subseteq R$ be a monomial ideal. Then,*

$$\ell_s(I) = \text{mdc}(\text{SP}(I)) + 1,$$

where $\text{mdc}(\text{SP}(I))$ denotes the maximum dimension of a compact face of $\text{SP}(I)$.

Example 2.12. Let $I = (xy, xz, yz)$ as in Example 2.6. It is known that

$$\text{SP}(I) = \text{convex hull} \left\{ (1, 1, 0), (1, 0, 1), (0, 1, 1), \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \right\} + \mathbb{R}_{\geq 0}^3.$$

It follows that the compact faces of maximum dimension of $\text{SP}(I)$ are the lines connecting $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and any of the other three vertices. Hence, the maximum dimension of a compact face of $\text{SP}(I)$ is 1. Therefore, by Proposition 2.11, $\ell_s(I) = 2$.

3. PROOF OF THEOREM 1.1

Let $I = (x, y)^{m_1} \cap (y, z)^{m_2} \cap (z, x)^{m_3}$ be the ideal of three fat coordinate points in $\mathbb{P}^2$. After permuting the variables if necessary, we may assume that $m_1 \geq m_2 \geq m_3 \geq 1$. By Proposition 2.5(2), the symbolic polyhedron $\text{SP}(I)$ of I is

$$\text{SP}(I) = \{(a, b, c) \in \mathbb{R}_{\geq 0}^3 \mid a + b \geq m_1, b + c \geq m_2, a + c \geq m_3\}.$$

We now examine the symbolic polyhedron $\text{SP}(I)$ of the ideal I , which describes the geometry of symbolic powers. The following theorem gives the explicit vertices of $\text{SP}(I)$.

Theorem 3.1. *The symbolic polyhedron $\text{SP}(I)$ has the following vertices:*

$$(0, m_1, m_3), \quad (m_1, 0, m_2),$$

and

1. $(m_1 - m_2, m_2, 0), (m_3, m_1 - m_3, 0)$ if $m_1 \geq m_2 + m_3$;
2. $(m_3, m_2, 0), (\frac{m_1+m_3-m_2}{2}, \frac{m_1+m_2-m_3}{2}, \frac{m_2+m_3-m_1}{2})$ if $m_1 \leq m_2 + m_3$.

Proof. Using a linear programming argument, each vertex of $\text{SP}(I)$ is the unique solution of a linear system whose equations are derived from the following set of equations:

$$a + b = m_1, \quad b + c = m_2, \quad a + c = m_3, \quad a = 0, \quad b = 0, \quad c = 0.$$

Thus, each vertex corresponds to the solution of a suitable system of three equalities, chosen from these six boundary hyperplanes. Since $m_1 \geq m_2 \geq m_3 \geq 1$, each such system contains at most one coordinate equation among $a = 0$, $b = 0$, and $c = 0$. Thus, we only need to consider the following four cases:

Case 1: Vertices on the coordinate face $a = 0$. If $(0, b, c)$ is a vertex of $\text{SP}(I)$, then $b \geq m_1, b + c \geq m_2, c \geq m_3$ and b and c are solutions of two of the following three equations: $b = m_1, b + c = m_2, c = m_3$. Since $m_1 \geq m_2 \geq m_3 \geq 1$, we have $b = m_1, c = m_3$. Thus, $(0, m_1, m_3)$ is a vertex of $\text{SP}(I)$.

Case 2: Vertices on the coordinate face $b = 0$. If $(a, 0, c)$ is a vertex of $\text{SP}(I)$, then $a \geq m_1, c \geq m_2, a + c \geq m_3$ and a and c satisfy two of the following three equations: $a = m_1, c = m_2, a + c = m_3$. Since $m_1 \geq m_2 \geq m_3 \geq 1$, it follows that $a = m_1, c = m_2$. Therefore, $(m_1, 0, m_2)$ is a vertex of $\text{SP}(I)$.

Case 3: Vertices on the coordinate face $c = 0$. Then the defining inequalities become

$$a + b \geq m_1, \quad b \geq m_2, \quad a \geq m_3.$$

Thus, on the plane $c = 0$, the symbolic polyhedron reduces to a two-dimensional polyhedron in the (a, b) -plane. Its vertices are obtained from the intersections of the boundary lines.

First, we consider the following system of equations:

$$\begin{cases} a + b = m_1, \\ b = m_2 \end{cases} \iff \begin{cases} a = m_1 - m_2, \\ b = m_2. \end{cases}$$

Hence, the point $(m_1 - m_2, m_2, 0)$ is feasible if and only if

$$a \geq m_3 \iff m_1 - m_2 \geq m_3 \iff m_1 \geq m_2 + m_3.$$

Second, we consider the following system of equations:

$$\begin{cases} a + b = m_1, \\ a = m_3 \end{cases} \iff \begin{cases} a = m_3, \\ b = m_1 - m_3. \end{cases}$$

Hence, the point $(m_3, m_1 - m_3, 0)$ is feasible if and only if

$$b \geq m_2 \iff m_1 - m_3 \geq m_2 \iff m_1 \geq m_2 + m_3.$$

Finally, we have a point $(m_3, m_2, 0)$. This point is feasible if and only if $m_2 + m_3 \geq m_1$.

Therefore, if $m_1 \geq m_2 + m_3$, then the additional vertices of $\text{SP}(I)$ are $(m_1 - m_2, m_2, 0)$ and $(m_3, m_1 - m_3, 0)$. Conversely, if $m_1 \leq m_2 + m_3$, then $\text{SP}(I)$ has one additional vertex $(m_3, m_2, 0)$.

Case 4: The interior vertex. Consider the intersection of the three planes

$$\begin{cases} a + b = m_1 \\ b + c = m_2 \\ a + c = m_3 \end{cases} \iff \begin{cases} a = \frac{m_1 + m_3 - m_2}{2} \\ b = \frac{m_1 + m_2 - m_3}{2} \\ c = \frac{m_2 + m_3 - m_1}{2}. \end{cases}$$

Hence, the point

$$\left(\frac{m_1 + m_3 - m_2}{2}, \frac{m_1 + m_2 - m_3}{2}, \frac{m_2 + m_3 - m_1}{2} \right)$$

is feasible if and only if $a, b, c \geq 0$. Since $m_1 \geq m_2 \geq m_3$, the latter condition is equivalent to

$$c \geq 0 \iff \frac{m_2 + m_3 - m_1}{2} \geq 0 \iff m_1 \leq m_2 + m_3.$$

In conclusion, the vertices of $\text{SP}(I)$ are always $(0, m_1, m_3), (m_1, 0, m_2)$, together with one of the following two cases:

1. $(m_1 - m_2, m_2, 0), (m_3, m_1 - m_3, 0)$ if $m_1 \geq m_2 + m_3$;
2. $(m_3, m_2, 0), \left(\frac{m_1 + m_3 - m_2}{2}, \frac{m_1 + m_2 - m_3}{2}, \frac{m_2 + m_3 - m_1}{2} \right)$ if $m_1 \leq m_2 + m_3$.

This completes the proof. □

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. By Proposition 2.7, the Waldschmidt constant $\widehat{\alpha}(I)$ is the minimum value of

$$a + b + c$$

subject to

$$(a, b, c) \in \text{SP}(I),$$

where

$$\text{SP}(I) = \left\{ (a, b, c) \in \mathbb{R}_{\geq 0}^3 \mid a + b \geq m_1, b + c \geq m_2, a + c \geq m_3 \right\}.$$

Since a linear function attains its minimum at a vertex of the feasible polyhedron, it suffices to evaluate the function

$$f(a, b, c) = a + b + c$$

at the vertices of $SP(I)$ described in Theorem 3.1.

First, we always have the vertices $(0, m_1, m_3)$, $(m_1, 0, m_2)$, and

$$f(0, m_1, m_3) = m_1 + m_3, \quad f(m_1, 0, m_2) = m_1 + m_2.$$

Case 1: $m_1 \geq m_2 + m_3$. In this case, the remaining vertices of $SP(I)$ are

$$(m_1 - m_2, m_2, 0), \quad (m_3, m_1 - m_3, 0).$$

Hence, $f(m_1 - m_2, m_2, 0) = m_1$, and $f(m_3, m_1 - m_3, 0) = m_1$. Therefore, $\hat{\alpha}(I) = m_1$. Since

$$m_1 \geq m_2, \quad m_1 \geq m_3, \quad m_1 \geq \frac{m_1 + m_2 + m_3}{2},$$

it follows that

$$\hat{\alpha}(I) = \max \left\{ m_1, m_2, m_3, \frac{m_1 + m_2 + m_3}{2} \right\}.$$

Case 2: $m_1 \leq m_2 + m_3$. In this case, the remaining vertices are $(m_3, m_2, 0)$, and

$$\left(\frac{m_1 + m_3 - m_2}{2}, \frac{m_1 + m_2 - m_3}{2}, \frac{m_2 + m_3 - m_1}{2} \right).$$

Thus, $f(m_3, m_2, 0) = m_2 + m_3$, while

$$f \left(\frac{m_1 + m_3 - m_2}{2}, \frac{m_1 + m_2 - m_3}{2}, \frac{m_2 + m_3 - m_1}{2} \right) = \frac{m_1 + m_2 + m_3}{2}.$$

Since $m_1 \leq m_2 + m_3$, we obtain $\frac{m_1 + m_2 + m_3}{2} \leq m_2 + m_3$. Hence,

$$\hat{\alpha}(I) = \frac{m_1 + m_2 + m_3}{2}.$$

Moreover,

$$\frac{m_1 + m_2 + m_3}{2} \geq m_1, \quad \frac{m_1 + m_2 + m_3}{2} \geq m_2, \quad \frac{m_1 + m_2 + m_3}{2} \geq m_3.$$

Therefore,

$$\hat{\alpha}(I) = \max \left\{ m_1, m_2, m_3, \frac{m_1 + m_2 + m_3}{2} \right\}.$$

This completes the proof. □

Example 3.2. We illustrate the two cases occurring in Theorem 1.1.

First, let

$$I_1 = (x, y)^7 \cap (y, z)^3 \cap (z, x)^2.$$

Here, $m_1 = 7$, $m_2 = 3$, and $m_3 = 2$, so that $m_1 \geq m_2 + m_3$. Therefore,

$$\hat{\alpha}(I_1) = \max \left\{ 7, 3, 2, \frac{7 + 3 + 2}{2} \right\} = 7.$$

By Theorem 3.1, the vertices of $SP(I_1)$ are

$$(0, 7, 2), \quad (7, 0, 3), \quad (4, 3, 0), \quad (2, 5, 0).$$

Hence, the minimum of $a + b + c$ at these vertices is 7.

Next, let

$$I_2 = (x, y)^6 \cap (y, z)^4 \cap (z, x)^3.$$

In this case, $m_1 = 6$, $m_2 = 4$, and $m_3 = 3$, so that $m_1 \leq m_2 + m_3$. Hence,

$$\widehat{\alpha}(I_2) = \max \left\{ 6, 4, 3, \frac{6+4+3}{2} \right\} = \frac{13}{2}.$$

The vertices of $\text{SP}(I_2)$ are

$$(0, 6, 3), \quad (6, 0, 4), \quad (3, 4, 0), \quad \left(\frac{5}{2}, \frac{7}{2}, \frac{1}{2} \right).$$

Thus, the minimum of $a + b + c$ is attained at

$$\left(\frac{5}{2}, \frac{7}{2}, \frac{1}{2} \right)$$

and is equal to $\frac{13}{2}$.

The first example illustrates the case in which the largest multiplicity determines the Waldschmidt constant, whereas the second illustrates the case in which it is determined by one half of the sum of the three multiplicities.

Remark 3.3. The formula in Theorem 1.1 can be used to obtain an explicit lower bound for the resurgence of I . Indeed, by the inequality of Bocci and Harbourne,

$$\frac{\alpha(I)}{\widehat{\alpha}(I)} \leq \rho(I),$$

the explicit value of $\widehat{\alpha}(I)$ obtained in Theorem 1.1, together with the initial degree $\alpha(I)$, provides a lower bound for $\rho(I)$. However, determining the exact value of the resurgence requires additional information about the ordinary powers of I and lies beyond the scope of the present paper.

We conclude the article by applying Theorem 3.1 to compute the symbolic analytic spread of the ideal defining three fat coordinate points.

Corollary 3.4. *Let $I = (x, y)^{m_1} \cap (y, z)^{m_2} \cap (z, x)^{m_3} \subseteq \mathbb{K}[x, y, z]$, where $m_1, m_2, m_3 \geq 1$. Then the symbolic analytic spread of I is $\ell_s(I) = 2$.*

Proof. We determine the compact faces of maximal dimension of $\text{SP}(I)$.

Case 1: $m_1 \geq m_2 + m_3$. By Theorem 3.1, the vertices of $\text{SP}(I)$ are

$$(0, m_1, m_3), \quad (m_1, 0, m_2), \quad (m_1 - m_2, m_2, 0), \quad (m_3, m_1 - m_3, 0).$$

The compact faces of maximal dimension are the line segments joining two vertices $(m_3, m_1 - m_3, 0)$ and $(m_1 - m_2, m_2, 0)$. Hence,

$$\text{mdc}(\text{SP}(I)) = 1.$$

Case 2: $m_1 \leq m_2 + m_3$. In this case, the vertices of $\text{SP}(I)$ are

$$(0, m_1, m_3), \quad (m_1, 0, m_2), \quad (m_3, m_2, 0), \quad \left(\frac{m_1 + m_3 - m_2}{2}, \frac{m_1 + m_2 - m_3}{2}, \frac{m_2 + m_3 - m_1}{2} \right).$$

The compact faces of maximal dimension are the line segments joining

$$\left(\frac{m_1 + m_3 - m_2}{2}, \frac{m_1 + m_2 - m_3}{2}, \frac{m_2 + m_3 - m_1}{2} \right)$$

to each of the other three vertices. Therefore, $\text{mdc}(\text{SP}(I)) = 1$.

Consequently, in both cases, $\ell_s(I) = \text{mdc}(\text{SP}(I)) + 1 = 2$ by Proposition 2.11. This completes the proof. $\square$

CONCLUSION

In this paper, we determined the vertices of the symbolic polyhedron associated with the ideal defining three fat coordinate points in $\mathbb{P}^2$. This description yields an explicit formula for the Waldschmidt constant and shows that the symbolic analytic spread of the ideal is equal to 2. Our arguments illustrate how these asymptotic invariants can be studied through polyhedral optimization.

A natural direction for further research is to extend this approach to ideals defining fat coordinate points in higher dimensional projective spaces. In this broader setting, it would be interesting to determine explicit formulas or effective bounds for the Waldschmidt constant, the symbolic analytic spread, the resurgence, and other related algebraic invariants.

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HẰNG SỐ WALDSCHMIDT CỦA BA ĐIỂM BÉO TỌA ĐỘ TRONG MẶT PHẪNG XẠ ẢNH $\mathbb{P}^2$

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TÓM TẮT

Hằng số Waldschmidt, được Waldschmidt giới thiệu vào năm 1975, đã trở thành một bất biến quan trọng trong đại số giao hoán và hình học đại số. Phần lớn các nghiên cứu hiện có tập trung vào việc thiết lập các chặn và khảo sát tính chất tiệm cận của hằng số này. Tuy nhiên, việc tính toán tường minh hằng số Waldschmidt vẫn còn khó khăn, ngay cả đối với những cấu hình điểm tương đối đơn giản trong mặt phẳng xạ ảnh. Trong bài báo này, chúng tôi xác định hằng số Waldschmidt và độ trải giải tích symbolic của idêan định nghĩa ba điểm béo tọa độ trong mặt phẳng xạ ảnh $\mathbb{P}^2$. Phương pháp của chúng tôi dựa trên việc mô tả đa diện symbolic của idêan và quy việc tính hằng số Waldschmidt của nó về một bài toán quy hoạch tuyến tính trên đa diện này.

Từ khóa: Điểm béo; đa diện symbolic; độ trải giải tích symbolic; lũy thừa symbolic; hằng số Waldschmidt.